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The Jespers–Van Oystaeyen Conjecture

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INTRODUCTION

Noncommutative Krull rings have been studied in some recent papers, cf., e.g. [3, 8, 12]. The mother examples of all these generalizations were studied already in 1968 by Fossum [6], i.e., maximal orders over Krull domains. One of the main problems in the theory is to find a suitable generalization of unique factorization domains and, related to this question, to find a proper definition of a class group.

Several possible definitions were suggested, e.g., the K-theoretic classgroup, $W(\Lambda)$, by Fossum [6], the normalizing classgroup, $Cl(\Lambda)$, by Chamarie [3] and the central classgroup, which has been studied extensively by Jespers and Wouters [7, 8].

Van Oystaeyen [22] and later Jespers [7] asked whether for a maximal order Λ over a Dedekind domain R; the vanishing of the central classgroup implies that Λ is an Azumaya algebra over R. This conjecture is readily checked to be equivalent with the following:

(*Jespers–Van Oystaeyen conjecture*). *If Λ is a maximal order over a Krull domain R, equivalent are*:

(1) *Λ is a reflexive Azumaya algebra in the sense of Orzech* [14].

(2) $Cl^c(\Lambda) \simeq Cl(R)$.

The aim of this paper is to show that this conjecture is virtually always satisfied. In the first section we treat the local case, i.e., maximal orders over a discrete valuation ring. If $Cl^c(\Lambda) = 1$ then either Λ is an Azumaya algebra or $Z(\Lambda/\Lambda \cdot m)$ is a purely inseparable field extension of $R/R \cdot m$. This result makes it possible to reduce the study to maximal orders over strict Henselian discrete valuation rings. An example of Saltman [18] is given to show that the conjecture is not true in general. In a forthcoming paper we aim to describe the possible exceptions by means of their universal measuring

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bialgebras [11]. These counterexamples force us to impose some extra condition on the maximal order. In this paper we will restrict attention to so called tamifiable maximal orders. In the second section we prove the conjecture for this class of orders. If Λ is moreover a flat module over its center, $Cl^c(\Lambda) \simeq Cl(R)$ entails that Λ is a (real) Azumaya algebra. Due to some counterexamples of Hoobler, the flatness condition cannot be dropped. Using the theory of generalized Rees rings (initiated by Van Oystaeyen [23] and extended to the noncommutative case in [11]) we will apply these results in the last section in order to reduce the study of maximal orders over a Krull domain R to the study of (reflexive) Azumaya algebras over certain Rees-type extensions $R(\Phi)$ of R.

The author is convinced that a further development of this approach will lead to a better understanding of maximal orders.

1. The Local Case

Throughout this paper, Λ will denote a maximal order over a Krull domain R, Σ will be the classical ring of quotients of Λ which is a central simple algebra over K, the field of fractions of R.

From [6] we retain that the set of all divisorial Λ-ideals (i.e., fractional ideals A such that $(A : \Lambda) : \Lambda = A$, cf. [6]) is the free Abelian group generated by the height one prime ideals of Λ, $\mathscr{D}(\Lambda)$. With $\mathscr{P}^c(\Lambda)$ we denote the subgroup of $\mathscr{D}(\Lambda)$ consisting of those divisorial ideals which are generated by one central element. The *central classgroup* of Λ, $Cl^c(\Lambda)$, is defined to be the quotient group $\mathscr{D}(\Lambda)/\mathscr{P}^c(\Lambda)$ cf., e.g. [7, 8].

In this first section we will restrict attention to the case that Λ is a maximal order over a discrete valuation ring R such that $Cl^c(\Lambda) = 1$. This means that the unique maximal ideal M of Λ is of the form $\Lambda \cdot m$ where m is the uniformizing parameter of R. In order to check that Λ is an Azumaya algebra over R, it is sufficient to check that $\Lambda/\Lambda \cdot m$ is a separable algebra over R/Rm [2].

The condition which appears in the literature, cf., e.g. [16], is the rather trivial one that $Z(\Lambda/\Lambda \cdot m)$ is a separable field extension of R/Rm (for, $\Lambda/\Lambda \cdot m$ is a simple p.i.-ring whence separable over its center). The aim of this section is to improve this result.

Let L be a separable splitting subfield of Σ and let S be the integral closure of R in L.

Theorem 1. *If Λ is a maximal order over a discrete valuation ring R, equivalent are*

(1) *Λ is an Azumaya algebra over R,*

(2) *$Cl^c(\Lambda) = 1$ and $\Lambda \otimes S$ is an* HNP-*ring.*

This theorem follows immediatly from the next two propositions. In the first proposition we aim to improve some results of Reiner and Riley and to reduce our study to two cases. The proof relies heavily on some results of McConnell [5] and Chamarie [3].

PROPOSITION 2. *If Λ is a maximal order over a discrete valuation ring R with $Cl^c(\Lambda)=1$, then one of the following situations occurs*:

(a) $Z(\Lambda/\Lambda \cdot m)=R/R \cdot m$ *in which case Λ is an Azumaya algebra*,

(b) $Z(\Lambda/\Lambda \cdot m)$ *is a purely inseparable field extension of* $R/R \cdot m$.

Proof. The proof will be split up in several steps:

Step 1. First, we claim that it is sufficient to check that prime ideals of the polynomial ring $\Lambda[t]$ which lie over $\Lambda \cdot m$ satisfy the unique-lying-over property with respect to the center $R[t]$. For, it is rather easy to see that this set of prime ideals corresponds bijectively to $\mathrm{Spec}(\Lambda/\Lambda \cdot m[t])$. Now, $\Lambda/\Lambda \cdot m$ is a simple p.i.-algebra, whence there is a one-to-one correspondence between $\mathrm{Spec}(\Lambda/\Lambda \cdot m[t])$ and $\mathrm{Spec}(Z(\Lambda/\Lambda \cdot m)[t])$. If the claimed condition is satisfied, this entails that there is a one-to-one correspondence between $\mathrm{Spec}(Z(\Lambda/\Lambda \cdot m)[t])$ and $\mathrm{Spec}(R/R \cdot m[t])$, i.e., there are no irreducible polynomials over R/Rm which decompose over $Z(\Lambda/\Lambda \cdot m)$ in distinct irreducible polynomials. Because $Z(\Lambda/\Lambda \cdot m)$ is a finite field extension of R/Rm this entails that $Z(\Lambda/\Lambda \cdot m)$ cannot contain separable elements over R/Rm not belonging to R/Rm, finishing the proof of our claim.

Step 2. In [3], Chamarie proved that a prime ideal P of a maximal order over a Krull domain satisfies the unique-lying-over property with respect to its center if and only if $\mathscr{C}(P)$, the multiplicatively closed set of elements which are regular modulo P, satisfies the left and right Öre-conditions. Let us first verify that every $P \in \mathrm{Spec}\,\Lambda[t]$ such that $P \cap \Lambda = \Lambda \cdot m$ satisfies the AR-property. By [5], 2.7 it is sufficient that P has a centralizing set of generators. Now, $m \in P$ and $P/\Lambda \cdot m[t] = \Lambda/\Lambda \cdot m[t] \cdot c'$ for some c' in $Z(\Lambda/\Lambda \cdot m)[t]$, because every ideal in a polynomial ring over a simple ring is generated by a central element. So, (m, c) is a centralizing set of generators of P. Using [5] Theorem 6 and Corollary 7, it will now be sufficient to check that every ideal of $\Lambda[t]$ has a centralizing set of generators. In fact, the proof of [5, Theorem 6] uses only the fact that the ideals H_n have a centralizing set of generators, so we just have to check this property for ideals intersecting Λ nontrivially.

Step 3. Let I be any ideal of $\Lambda[t]$ such that $I \cap \Lambda \neq 0$, then $I \cap \Lambda = \Lambda \cdot m^n$ for some natural number n. Let $I_1 = \mu_1(I)$ where $\mu_1 \colon \Lambda[t] \to \Lambda[t]/(m^n)$ is the canonical epimorphism and let $c_1 \in I$ be of minimal degree such that $\mu_1(c_1) \neq 0$. If m_1 is the leading coefficient of c_1, then clearly $\mu_1(m_1) \neq 0$ and

$\Lambda \cdot m_1 \cdot \Lambda = \Lambda \cdot m^{l_1}$ where $l_1 < n$, for, otherwise one could lower the degree of c_1. So, we may suppose that the leading coefficient of c_1 equals m^{l_1}. Because $m^{l_1} \in R$ and the degree of c_1 is minimal, $c_1\lambda - \lambda c_1 \in (m^n)$ for every $\lambda \in \Lambda$ yielding that $\mu_1(c_1) \in Z(\Lambda[t]/(m^n))$. If $l_1 = 0$ (i.e., $m_1 = 1$) then $\mu_1(I) = \Lambda[t]/(m^n) \cdot \mu_1(c_1)$, finishing the proof. If $0 < l_1 < n$ and if $I \neq (m^n, c_1)$, choose $c_2 \in I$ of minimal degree such that $\mu_2(c_2) \neq 0$ where $\mu_2 : \Lambda[t] \to \Lambda[t]/(m^n, c_1)$ is the canonical epimorphism. Clearly, by a minimal degree argument as before we may assume that the leading coefficient of c_2 equals m^{l_2} for some $l_2 < l_1$ and that $c_2\lambda - \lambda c_2 \in (m^n, c_1)$ for every $\lambda \in \Lambda$ whence $\mu_2(c_2) \in Z(\Lambda[t]/(m^n, c_1))$. Continuing in this manner one finds after a finite number of times an element c_m such that either $I = (m^n, c_1, ..., c_m)$ or the leading coefficient of c_{m+1} is 1 yielding that $I = (m^n, c_1, ..., c_{m+1})$, finishing the proof.

The condition: $Z(\Lambda/\Lambda \cdot m)$ is not a purely inseparable field extension of $R/R \cdot m$, is always satisfied in the cases under consideration in algebraic number theory and algebraic geometry. For, in these cases, Σ is a central simple algebra over a global field or over a functionfield of a variety over a basefield of characteristic zero, yielding that $R/R \cdot m$ is a perfect field.

This vast amount of good examples may account for the manifest lack of interest of order-theorists in the question whether there exist maximal orders satisfying condition (b) of Proposition 2. Despite this indifference we will prove, just for the sake of aesthetics:

PROPOSITION 3. *If Λ is a maximal order over a discrete valuation ring R with $Cl^c(\Lambda) = 1$ such that $\Lambda \otimes S$ is an* HNP-*ring, then situation* (b) *cannot occur.*

Proof. Again, we divide the proof in three steps:

Step 1. Suppose that $Z(\Lambda/\Lambda \cdot m)$ is a proper purely inseparable field extension of $R/R \cdot m$. By a result of [9] we know that the natural map between the Brauer-groups:

$$[- \otimes Z(\Lambda/\Lambda \cdot m)] : \mathrm{Br}(R/R \cdot m) \to \mathrm{Br}(Z(\Lambda/\Lambda \cdot m))$$

is an epimorphism. So, there exists a central simple algebra A over $R/R \cdot m$ such that $M_k(A) \otimes Z(\Lambda/\Lambda) \simeq M_l(\Lambda/\Lambda \cdot m)$.

Replacing Λ by $M_l(\Lambda)$, Σ by $M_l(\Sigma)$, A by $M_k(A)$ etc. we may therefore assume that $\Lambda/\Lambda \cdot m$ contains a simple algebra A over $R/R \cdot m$ such that $\Lambda/\Lambda \cdot m \simeq A \otimes Z(\Lambda/\Lambda \cdot m)$. Now, if $\mu : \Lambda \to \Lambda/\Lambda \cdot m$ denotes the natural epimorphism we will denote by $A_1 = \mu^{-1}(A)$. Because A_1 and Λ share the common twosided ideal $\Lambda \cdot m$, A_1 is an order in Σ and the center of A_1 equals R. Furthermore, $\Lambda \cdot m$ is the unique nonzero prime ideal of A_1 and $Z(A_1/\Lambda \cdot m) = R/R \cdot m$. For order purists, A_1 is a Bäckström-order [17] with associated hereditary order Λ.

Step 2. Let L be a separable splitting field for Σ contained in Σ. Further, let S be the integral closure of R in L. It is fairly easy to check that $S = L \cap \Lambda$ is a discrete valuation ring with uniformizing parameter m (this follows, e.g., from the theory of Van Geel primes and their extension theorem [19]). Now, $\Lambda \otimes S$ is by assumption an hereditary order in $M_n(L)$ which is not maximal because otherwise Λ would be Azumaya (cf. [17, Theorem VI.2.8] or an easy descent argument). Now, by results of Harada or Artin [1] one can describe $\Lambda \otimes S$ nicely in the following way

$$\begin{pmatrix} M_{n_1}(S) & S_{n_1 \times n_2} & \cdots & S_{n_1 \times n_j} \\ m \cdot S_{n_2 \times n_1} & M_{n_2}(S) & \cdots & S_{n_2 \times n_j} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ m \cdot S_{n_j \times n_1} & m \cdot S_{n_j \times n_2} & \cdots & M_{n_j}(S) \end{pmatrix}$$

with $n_1 + n_2 + \cdots + n_j = n$ and $j \geqslant 2$. Clearly, $A_1 \otimes S$ shares a common ideal with $\Lambda \otimes S$, namely, $(\Lambda \otimes S) \cdot m$, therefore

$$(\Lambda \otimes S) \cdot m \subset A_1 \otimes S \subset \Lambda \otimes S.$$

This implies that there are at least j prime ideals of $A_1 \otimes S$ lying over mS. The proof will be complete if we can show that this is not possible.

Step 3. Because $\xi: A_1 \to A_1 \otimes S$ is a central extension, prime ideals intersect in prime ideals, so we have to calculate the fiber of ξ in $\Lambda \cdot m$. A_1 being a finite module over its center and $\Lambda \cdot m$ satisfying the unique-lying-over property with respect to the center, $\Lambda \cdot m$ is localizable [22] whence there is a one-to-one correspondence between this fiber and $\operatorname{Spec}(A \otimes S) = \operatorname{Spec}(S/S \cdot m)$ because A is a simple algebra with center $R/R \cdot m$; $S/S \cdot m$ being a field finishes the proof.

The inverse implication of Theorem 1 is, of course, trivial. We will end this section with the presentation of an ètàle approach to the problem. In particular, we will reduce the Jespers–Van Oystaeyen conjecture to the special case of a maximal order over a strict Henselian discrete valuation ring:

THEOREM 4. *If Λ is a maximal order over a discrete valuation ring R and if R^{sh} denotes the strict Henselization of R then the following statements are equivalent*:

(1) *Λ is an Azumaya algebra over R*,

(2) *$Cl^c(\Lambda) = 1$ and R^{sh} splits Σ.*

Proof. (1) $\Rightarrow$ (2). Trivial because the Brauer group of a strict Henselian local ring is trivial, i.e., R^{sh} splits Λ, hence Σ.

(2) $\Rightarrow$ (1). First, we recall that R^{sh} is a discrete valuation ring with uniformizing parameter m, $R^{sh}/R^{sh} \cdot m$ is the separable closure of $R/R \cdot m$ and $R \to R_{sh}$ is an ètàle extension, hence in particular a Galois extension in the sense of Chase, Harrisson and Rosenberg [4]. We claim that $\Lambda \otimes R^{sh}$ is an HNP-ring (for, $J(\Lambda \otimes R^{sh}) = J$ considered as a Λ-module, denoted J_Λ, is f.g. projective and using the separability idempotent of R^{sh} it is easy to check that the natural map $J_\Lambda \otimes R^{sh} \to J$ splits whence J is a f.g. projective $\Lambda \otimes R^{sh}$-module). By assumption R^{sh} splits Σ whence $\Lambda \otimes R^{sh}$ is of the form as described in the proof of Proposition 3 so it will be sufficient to check that the fiber of $\Lambda \otimes R^{sh} \leftarrow R^{sh}$ at $R^{sh} \cdot m$ consists of one element (by a descent argument). Now, this fiber is in one-to-one correspondence with $\mathrm{Spec}(\Lambda/\Lambda \cdot m \otimes R^{sh}/R^{sh} \cdot m) \simeq \mathrm{Spec}(Z(\Lambda/\Lambda \cdot m) \otimes R^{sh}/R^{sh} \cdot m)$. By Proposition 2 $Z(\Lambda/\Lambda \cdot m)$ is purely inseparable over $R/R \cdot m$ and $R^{sh}/R^{sh} \cdot m$ is separable over $R/R \cdot m$ whence this set consists of one element, done.

Remark 5. (a) By an argument as in the foregoing proof, $\Lambda \otimes R^{sh}$ is always an HNP-ring with a unique non-zero prime ideal which is centrally generated. Therefore, $\Lambda \otimes R^{sh}$ is a maximal order over R^{sh} with trivial central classgroup. Thus, the Jespers–Van Oystaeyen conjecture holds for maximal orders over R if and only if it holds for maximal orders over R^{sh}.

(b) Up till now, we have reduced the original question to the following one

Does there exists a discrete valuation ring Λ (in a p.i. skewfield) with central uniformizing parameter such that its center R is a strict Henselian discrete valuation ring and $\Lambda/\Lambda \cdot m$ is a commutative (!) *purely inseparable field extension of $R/R \cdot m$?*

For, $\Lambda/\Lambda \cdot m$ is a division ring over its center, but since $\mathrm{Br}(R/R \cdot m) \to \mathrm{Br}(Z(\Lambda/\Lambda \cdot m))$ is epimorphic and $\mathrm{Br}(R/R \cdot m) = 1$ (R being a strict Henselian valuation ring), $\Lambda/\Lambda \cdot m = Z(\Lambda/\Lambda \cdot m)$.

Let us recall an example due to Saltman [18] which shows that such a situation can occur:

Let F be a field of characteristic p and $K = F((t))$, the field of Laurent sequences over F equipped with the natural discrete valuation and let R be the associated (complete) valuation ring. Let $\{a, b\}$ be contained in a p-basis for F (e.g., over its prime field) and let Δ be the cyclic algebra $[at^{-p}, b)$. Now, choose $\alpha \in \Delta$ such that $\alpha^p - \alpha = a \cdot t^{-p}$ then $(\alpha \cdot t)^p - t^{p-1}(\alpha \cdot t) = a$ whence $K(\alpha)/K$ is a field extension such that the corresponding residue fields are $F(a^{1/p})$ and F. Since $b \notin (F(a^{1/p}))^p$, one can verify that b is not a norm of $K(\alpha)/K$ yielding that Δ is a skewfield. Since any valuation on a complete field extends to a finite dimensional skewfield over it, there exists a valuation ring Λ in Δ over R with $Cl^c(\Lambda)=1$ and one easily verifies that $\Lambda/\Lambda\cdot t=F(a^{1/p},b^{1/p})$.

In a subsequent paper we aim to generalize Saltman's approach (only for exponent one and degree p^2-extensions) and combine it with the theory of universal bialgebras associated with orders (as expounded by the author in [10]).

2. The Global Case

If Λ is a maximal order over a Krull domain R, then there is a natural morphism $\eta\colon Cl(R)\to Cl^c(\Lambda)$ induced by the morphism $\Theta\colon\mathscr{D}(R)\to\mathscr{D}(\Lambda)$ defined by $\Theta(A)=(\Lambda\cdot A)^{**}$. In this section we aim to investigate to what extend $Cl(R)\simeq Cl^c(\Lambda)$ implies that Λ is an Azumaya algebra over R. Let us first recall some definitions:

If Λ is an order over a Krull domain R, consider the natural R-algebra morphism:

$$m\colon\Lambda^e=\Lambda\otimes\Lambda^{\mathrm{opp}}\to\mathrm{End}_R(\Lambda)$$

which is defined by $m(\sum a_i\otimes b_i)(\lambda)=\sum a_i\cdot\lambda\cdot b_i$. If Λ is a divisorial R-lattice (i.e., $\Lambda=\Lambda^{**}$), so is $\mathrm{End}_R(\Lambda)$. This entails that m extends to a homomorphism m' from $(\Lambda\otimes\Lambda^{\mathrm{opp}})^{**}$ to $\mathrm{End}_R(\Lambda)$:

$$\begin{tikzcd}
\Lambda\otimes\Lambda^{\mathrm{opp}} \arrow[r] \arrow[d] & \mathrm{End}_R(\Lambda) \\
(\Lambda\otimes\Lambda^{\mathrm{opp}})^{**} \arrow[ur, "m'"'] &
\end{tikzcd}$$

Extending an idea of Yuan [25], Orzech defines in [14] a reflexive Azumaya algebra Λ over a Krull domain R to be an R-algebra which is a divisorial R-lattice such that the morphism m' defined above is an isomorphism.

Two reflexive Azumaya algebras Λ and Γ are said to be similar if there exist divisorial R-lattices M and N such that

$$(\Lambda\otimes\mathrm{End}_R(M))^{**}\simeq(\Gamma\otimes\mathrm{End}_R(N))^{**}.$$

The similarity classes of reflexive Azumaya algebras over R form a group $\beta(R)$, the so called reflexive Brauer group.

The next lemma is due to Van Oystaeyen (even in a more general setting, [22]):

Lemma 6. *If Λ is a reflexive Azumaya algebra over a Krull domain R, then Λ is a maximal order.*

Proof. Suppose that Λ is properly contained in a maximal R-order Γ. Because Λ_p is an Azumaya algebra for every $p \in X^{(1)}(R)$, the set of minimal non-zero prime ideals of R, $\Lambda_p = \Gamma_p$ yielding that Γ/Λ is σ—torsion where $\sigma = \inf\{\sigma_p; p \in X^{(1)}(R)\}$. Because Λ is σ closed (being divisorial!) this entails that $\Lambda = \Gamma$, a contradiction.

An order Λ over a Krull domain R is said to be *tame* if it is a divisorial R-lattice and if Λ_p is an HNP-ring for every $p \in X^{(1)}(R)$. Again, let L be a separable splitting subfield of Σ and let S be the integral closure of R in L. S is of course again a Krull domain. We are now able to state the main theorem of this paper:

THEOREM 7. *If Λ is a maximal order over a Krull domain R, then:*

(a) *Λ is a reflexive Azumaya algebra if and only if $Cl(R) = Cl^c(\Lambda)$ and $\Lambda \otimes S$ is a tame order.*

(b) *Λ is an Azumaya algebra if and only if $Cl(R) = Cl^c(\Lambda)$, $\Lambda \otimes S$ is a tame order and Λ is a flat R-module.*

Proof. (a) Recall from Lemma 6 that every (reflexive) Azumaya algebra is indeed a maximal order. For every maximal order Λ over a Krull domain R, we have the exact diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathscr{P}(R) & \longrightarrow & \mathscr{D}(R) & \longrightarrow & Cl(R) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \\
1 & \longrightarrow & \mathscr{P}^c(\Lambda) & \longrightarrow & \mathscr{D}(\Lambda) & \longrightarrow & Cl^c(\Lambda) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & \bigoplus \mathbb{Z}/n_p\mathbb{Z} & & & &
\end{array}$$

Coker(μ) is a finite group. For, take any element in the Formanek-center, cf. [15] then there are only a finite number of height one prime ideals P of Λ such that $\Lambda \cdot c \subset P$. Because the localizations at the other height one prime ideals are Azumaya-algebras, $P = \Lambda \cdot (P \cap R)$ for allmost all $P \in X^{(1)}(\Lambda)$. And, for the finitely many exceptions $P = (\Lambda \cdot (P \cap R)^{n_p})^{**}$.

Now, if $Cl(R) \simeq Cl^c(\Lambda)$, then every $P \in X^{(1)}(\Lambda)$ is centrally generated and therefore Λ_p is a maximal order over R_p with $Cl^c(\Lambda) = 1$ for every $p \in X^{(1)}(R)$. Because discrete valuations extend in a finite separable field extension, one can find for every $p \in X^{(1)}(R)$ an height one prime ideal P in S such that $(\Lambda \otimes S)_P \simeq \Lambda_p \otimes S_P$ is an HNP-ring (because $\Lambda \otimes S$ is a tame order). By Theorem 1 this implies that Λ_p is an Azumaya-algebra for every $p \in X^{(1)}(R)$.

To finish the proof we have to check that the morphism:

$$m' : (\Lambda \otimes \Lambda^{\text{opp}})^{**} \to \operatorname{End}_R(\Lambda)$$

is an isomorphism. Because m' is a morphism between two σ—closed R-modules, it is clearly sufficient to check that $(m')_p$ is an isomorphism for every $p \in X^{(1)}(R)$. But this is trivial because Λ_p is an Azumaya-algebra.

(b) In view of part (a) it suffices to prove that the morphism:

$$i : \Lambda \otimes \Lambda^{\text{opp}} \to (\Lambda \otimes \Lambda^{\text{opp}})^{**}$$

is an isomorphism. It is clearly monomorphic. To prove surjectivity, let $\alpha = \sum \lambda_i \otimes \mu_i / r \in \bigcap (\Lambda \otimes \Lambda^{\text{opp}})_p$, where $\lambda_i \in \Lambda$, $\mu_i \in \Lambda^{\text{opp}}$ and $r \in R$. Because R is a Krull domain, R satisfies the finite character property, i.e., $I = \{p \in X^{(1)}(R) : R \notin R_p^*\}$ is a finite set. Let $J = X^{(1)}(R) - I$, $\Gamma = \bigcap \{\Lambda_p^{\text{opp}} ; p \in I\}$ and $\Gamma' = \bigcap \{\Lambda_p^{\text{opp}} ; p \in J\}$. Then, $\alpha \in \Lambda \otimes \Gamma'$ and clearly $\alpha \in \bigcap \{(\Lambda \otimes \Lambda^{\text{opp}})_p ; p \in I\} = \Lambda \otimes \bigcap \{\Lambda_p^{\text{opp}} ; p \in I\}$ because I is finite and Λ^{opp} is a flat R-module. Therefore,

$$\alpha \in \Lambda \otimes \Gamma \cap \Lambda \otimes \Gamma' = \Lambda \otimes (\Gamma \cap \Gamma') = \Lambda \otimes \Lambda^{\text{opp}}$$

because Λ^{opp} is a divisorial R-lattice (as a maximal order), finishing the proof.

Remark 8. Now, suppose that every reflexive Azumaya algebra over R is a flat R-module, then this would entail that they are Azumaya, yielding that the natural map $\operatorname{Br}(R) \to \beta(R)$ is epimorphic. However, in general this is not the case as some counterexamples due to Hoobler show. This proves that the flatness-condition cannot be dropped.

3. A New Approach to Maximal Orders

In this section we aim to apply the foregoing results in order to reduce the study of maximal orders over a (nice) Krull domain R to:

(a) The study of graded (reflexive) Azumaya algebras over certain Rees-type extensions $R(\Phi)$ constructed from R (i.e. the study of the graded (reflexive) Brauer group of $R(\Phi)$, cf. [24]).

(b) The study of the ringextension $R \to R(\Phi)$.

Moreover this approach enables us to calculate the Brauer group of the field of fractions of R in terms of the graded (reflexive) Brauer group of the rings $R(\Phi)$ as well as to give a Brauer-group interpretation of the Jespers–Van Oystaeyen conjecture.

Throughout, Λ will be a maximal order over a Krull domain R and $\{P_1,..., P_n\}$ will be the finite number of height one prime ideals of Λ which are not centrally generated and $\operatorname{coker}(\mu) = \bigoplus \mathbb{Z}/n_i\mathbb{Z}$ (cf. proof of Theorem 7).

We consider the $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$-graded subring $\Lambda(\Phi)$ of $\Sigma[X_1, X_1^{-1},..., X_n, X_n^{-1}]$ which is defined by

$$\Lambda(\Phi)(m_1,..., m_n) = (P_1^{m_1} \cdots P_n^{m_n})^{**} \cdot X_1^{m_1} \cdots X_n^{m_n}$$

Part (a) of the next theorem is an adaptation of a similar result in [11], therefore we will merely present an outline of the proof.

THEOREM 9. *If Λ is a maximal order over a Krull domain R, then*:

(a) *$\Lambda(\Phi)$ is a p.i. maximal order over its center $R(\Phi)$ which is a Krull domain*,

(b) $Cl^c(\Lambda(\Phi)) \simeq Cl(R(\Phi))$.

Proof. (a) In view of [3] we have to check the following two facts:

1. For any ideal I of $\Lambda(\Phi)$, $(I :_l I) = (I :_r I) = \Lambda(\Phi)$.

2. $\Lambda(\Phi)$ satisfies the ACC on divisorial ideals (i.e. ideals of Λ satisfying $(I : \Lambda(\Phi)) : \Lambda(\Phi) = I$).

(1) Because $\Lambda(\Phi)$ is a graded p.i. ring, its graded ring of quotients, $Q^g(\Lambda(\Phi)) = \Sigma[X_i, X_i^{-1}]$ is obtained by inverting central homogeneous elements and it is an Azumaya algebra over the Krull domain $K[X_i, X_i^{-1}]$, cf. [11]. So, $\Sigma[X_i, X_i^{-1}]$ is a maximal order. Now, let I be any ideal of $\Lambda(\Phi)$ and suppose that $I \cdot q \subset I$ for some $q \in Q(\Lambda(\Phi))$. Then, $Q^g(\Lambda(\Phi)) \cdot I \cdot q \subset Q^g(\Lambda(\Phi)) \cdot I$ and by maximality of $Q^g(\Lambda(\Phi))$ this yields that $q \in Q^g(\Lambda(\Phi))$. Hence we may decompose q in its homogeneous components, $q = q_{i_1} + \cdots + q_{i_k}$ with $i_1 \leqslant \cdots \leqslant i_k$ (note that $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ can be given the structure of an ordered group. We obtain: $C_0(I) \cdot q_{i_k} \subset C_{i_k}(I)$ where $C_i(I)$ denotes the set of all leading coefficients of elements of I of degree i. Therefore, $(C_0(I) \cdot q_{i_k})^{**} \subset C_{i_k}(I)^{**}$ whence: $q_{i_k} \in (C_0(I)^{**})^{-1} * C_{i_k}(I)^{**}$. By [11], this means that $q_{i_k} \in \Lambda(\Phi)(i_k)$. Replacing q by $q - q_{i_k}$ and repeating the foregoing argumentation one finally arrives at $q \in \Lambda(\Phi)$, finishing the proof of (1).

(2) If $\{I_n ; n \in \mathbb{N}\}$ is an ascending chain of divisorial $\Lambda(\Phi)$ ideals, then the ascending chain $\{(Q^g \cdot I_n)^{**}; n \in \mathbb{N}\}$ becomes stationary, i.e. there is an $n' \in \mathbb{N}$ such that $(Q^g \cdot I_m)^{**} = (Q^g \cdot I_{n'})^{**}$ for every $m \geqslant n'$. On the other hand, because Λ is a maximal order, there exists an $n'' \in \mathbb{N}$ such that:

$C_0(I_m)^{**} = C_0(I_{n''})^{**}$ for every $m \geqslant n''$. Let $N = \sup(n', n'')$, then $I_m = I_N$ for every $m \geqslant N$, cf. [11].

(b) The graded central classgroup of $\Lambda(\Phi)$, $Cl_g^c(\Lambda(\Phi))$ is defined to be

$$Cl_g^c(\Lambda(\Phi)) = \mathscr{D}_g(\Lambda(\Phi))/\mathscr{P}_g^c(\Lambda(\Phi))$$

where $\mathscr{D}_g(\Lambda(\Phi))$ is the subgroup of $\mathscr{D}(\Lambda(\Phi))$ of the $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$-graded divisorial ideals of $\Lambda(\Phi)$ and $\mathscr{P}_g^c(\Lambda(\Phi)) = \{\Lambda(\Phi) \cdot c \mid c \in k[X_1, X_1^{-1},..., X_n, X_n^{-1}]\}$. By [8, Theorem 3.2] the sequence

$$1 \to Cl_g^c(\Lambda(\Phi)) \to Cl^c(\Lambda(\Phi)) \to Cl^c(\Sigma[X_1, X_1^{-1},..., X_n, X_n^{-1}]) \to 1$$

is exact. Now, $\Sigma[X_1, X_1^{-1},..., X_n, X_n^{-1}]$ being an Azumaya-algebra over a factorial domain, $Cl^c(\Sigma[X_1, X_1^{-1},..., X_n, X_n^{-1}]) = 1$ whence: $Cl_g^c(\Lambda(\Phi)) \simeq Cl^c(\Lambda(\Phi))$.

Furthermore, it is easy to verify that the sequence

$$1 \to \langle [P_1],..., [P_n] \rangle \to Cl^c(\Lambda) \to Cl_g^c(\Lambda(\Phi)) \to 1$$

is exact. Similarly, $Cl_g(R(\Phi)) \simeq Cl(R(\Phi))$ and

$$1 \to \langle [p_1],..., [p_n] \rangle \to Cl(R) \to Cl_g(R(\Phi)) \to 1$$

whence one obtains finally the exact diagram

$$\begin{tikzcd}
& 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \langle [p_i] \rangle \arrow[r] \arrow[d] & Cl(R) \arrow[r] \arrow[d] & Cl(R(\Phi)) \arrow[r] & 1 \\
1 \arrow[r] & \langle [P_i] \rangle \arrow[r] \arrow[d] & Cl^c(\Lambda) \arrow[r] \arrow[d] & Cl^c(\Lambda(\Phi)) \arrow[r] & 1 \\
& \bigoplus \mathbb{Z}/n_i \cdot \mathbb{Z} & \bigoplus \mathbb{Z}/n_i \cdot \mathbb{Z} & &
\end{tikzcd}$$

finishing the proof.

If the Jespers–Van Oystaeyen conjecture would be true in general, then the foregoing result completes our reduction from maximal orders over a Krull domain R to graded (reflexive) Azumaya algebras over suitable Rees-type extensions.

This is, in particular, the case for applications in algebraic geometry if the characteristic of the ground field is zero.

In general, we will prove that the foregoing reduction holds also if the maximal order Λ over R is *tamifiable*. By this we mean that $\Lambda \otimes S$ is a tame order, where S is the integral closure of R in some separable splitting subfield L of Σ. We have to prove the following:

THEOREM 10. *If Λ is tamifiable, hence so is $\Lambda(\Phi)$.*

Proof. Of course, $L(X_1,..., X_n)$ is a separable splitting subfield of $\Sigma(X_1,..., X_n)$. Let $S(\Phi)$ be the integral closure of $R(\Phi)$ in $L(X_1,..., X_n)$. Because $R(\Phi)$ is a graded Krull domain, so is $S(\Phi)$ by an argument similar to [20]. Let P be any height one prime ideal of $S(\Phi)$, then either P is a graded prime ideal or P_g (the set of homogeneous elements) $= 0$.

Suppose first that $P_g = 0$. Then the localization of $\Lambda(\Phi) \otimes S(\Phi)$ at P is a localization of $\Sigma \otimes L[X_1, X_1^{-1},..., X_n, X_n^{-1}]$. Therefore it will be an Azumaya algebra over the Krull domain $S(\Phi)_P$; hence a maximal (thus tame) order.

Next, suppose that P is a graded prime ideal and that $P \cap R = p$. If $p \notin \{P_1 \cap R,..., P_n \cap R\}$, then the localization of $\Lambda(\Phi) \otimes S(\Phi)$ at P is a localization of $(\Lambda_p \otimes S_{P\cap S})[X_1, X_1^{-1},..., X_n, X_n^{-1}]$ whence a tame order because the class of tame orders is closed under polynomial extensions and central localizations [6].

If $p = P_1 \cap R$, then $(\Lambda(\Phi) \otimes S(\Phi))_P = (\Lambda(\Theta) \otimes S(\Theta))_q[X_2, X_2^{-1},..., X_n, X_n^{-1}]$ where $\Lambda(\Theta)(n) = P_1^n \cdot X_1^n$, $S(\Theta)$ is the integral closure of $R(\Theta)$, the center of $\Lambda(\Theta)$, in $L(X_1)$ and $q = P \cap S(\Theta)$. Now, $\Lambda(\Theta) \otimes S(\Theta)$ is readily checked to be an overring of $(\Lambda \otimes S)(\Phi)$ in $(\Sigma \otimes L)(X_1)$. Furthermore, $(\Lambda \otimes S)(\Phi)$ is a tame order by [11] or [13] and therefore so is $\Lambda(\Theta) \otimes S(\Theta)$, finishing the proof.

To end this paper we will present two Brauer group interpretations of the Jespers–Van Oystaeyen conjecture. The proof and more details will appear elsewhere.

All rings $R(\Phi)$ occuring as centers of generalized Rees rings of maximal orders over R are of the following type:

Let R be a Krull domain, then for any (finite) set of height one prime ideals $\{p_1,..., p_n\}$ and any set of natural numbers $\{m_1,..., m_n\}$ we will define the so called lepidopterous Rees ring (in the terminology of Van Oystaeyen) $R(p_i, m_i)$ to be the $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$-graded ring

$$R(p_i, m_i)(i_1,..., i_n) = (p_1^{[[i_1/m_1]]} \cdots p_n^{[[i_n/m_n]]})^{**}$$

where $[[a/b]] = \mathrm{sign}(a/b) \cdot [|a/b|]$, ($[\cdot/\cdot]$ denotes the integral part of $\cdot/\cdot$). Mimicing the proofs of Theorems 4.5 and 4.7 of [11] it is fairly easy to see that all these rings $R(p_i, m_i)$ are again Krull domains.

They are ordered in the following way: $R(p_i, m_i) \leqslant R(p'_j, m'_j)$ if and only if $\{p_i\} \subset \{p'_j\}$ and for the corresponding values of i and j: $m_i \mid m'_j$.

For more details on graded (reflexive) Brauer groups, the reader is referred to [24].

THEOREM 11. (1) *If R is a Krull domain with field of fractions K and if the Jespers–Van Oystaeyen conjecture holds for all $R(p_i, m_i)$, then* $Br(K) = \lim(\beta^g(R(p_i, m_i)))$.

(2) *Moreover, if R is a Dedekind domain then* $Br(K) = \lim(Br^g(R(p_i, m_i)))$.

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